Isomorphism of pointed minimal systems is not classifiable by countable structures

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Borel reduction

- Let E and F be two equivalence relations on Polish spaces X and Y, respectively.
- We say E is **Borel reducible** to F if there exists a Borel function $f : X \to Y$ such that

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

Denoted by $E \leq_B F$.
We will regrad F as a more complicated equivalence relation.

Benchmarks

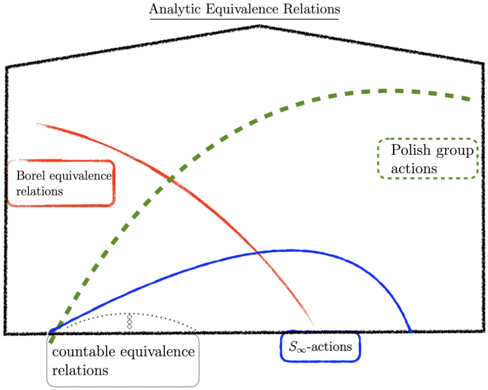


Figure: The Zoo

- The Main purpose of this area is to place equivalence relations arising in dynamical systems into this picture.

Equivalence relations which are not classifiable by countable structures

- Let S_∞ be the permutation group of natural numbers. This group and its subgroup have serious connections with countable model theory.

Definition

An Equivalence relation E is classifiable by countable structures if there exists a Borel S_∞ action $E_X^{S_\infty}$ such that $E \leq_B E_X^{S_\infty}$.

Maximal S_∞ actions

The following four equivalence relations are maximal S_∞ actions, in other words, all S_∞ actions are Borel reducible to it.

- Isomorphism of countable graphs.
- (Carmelo, Gao) Homeomorphism relation of zero-dimensional compact metric spaces.
- (Carmelo, Gao) Conjugacy relation of Cantor systems.
- (Paolini, Shelah) Isomorphism of Torsion-free abelian groups.

Turbulent actions

(Hjorth) Let G be a Polish group acting on a Polish space X Borelly, suppose we have

1. Every orbit is meager.
2. Every orbit is dense.
3. Every local orbit is somewhere dense.

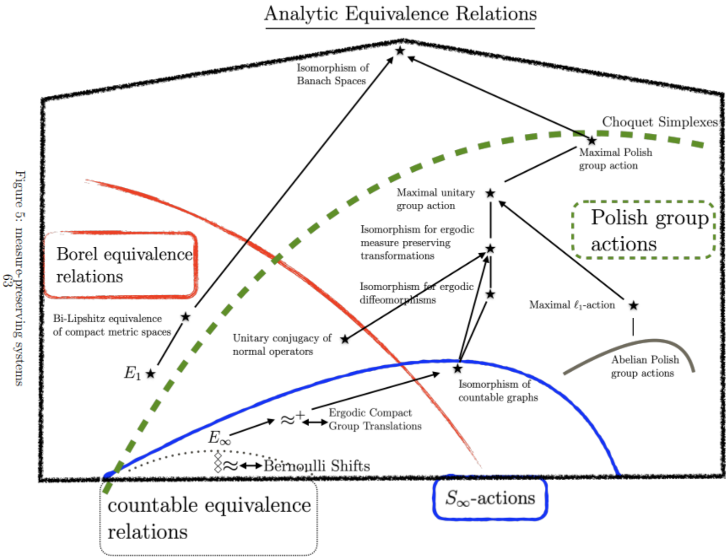
Then, E_X^G is not reducible to any Borel S_∞ action. We call such an action a **turbulent action**.

Some known results in dynamical systems

- (Hjorth) Isomorphism of ergodic measure preserving transformations is not classifibile by countable structures.
- (Foreman, Weiss) The conjugation action of measure preserving transformations of $[0, 1]$ on the space of ergodic measure preserving transformations of $[0, 1]$ is turbulent.
- (Foreman, Rudolph, Weiss) The conjugacy relation of ergodic measure preserving transformations of $[0, 1]$ is not Borel.
- (Foreman, Gorodetski) Topological conjugacy relation of diffeomorphism of a smooth manifold with dimension at least five is not Borel.

  Exact complexity of the two equivalence relations above are open.

New graph


Analytic Equivalence Relations
Isomorphism of Banach Spaces
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Unitary conjugacy of normal operators
Maximal ℓ_1-action
Abelian Polish group actions
E_1
Isomorphism of countable graphs
$\approx^+$ Ergodic Compact Group Translations
E_∞
$\approx$ Bernoulli Shifts
countable equivalence relations
S_∞-actions
Figure 5: measure-preserving systems
63

Minimal system

A **compact system** is a compact metric space together with an automorphism f, (X, f).
A **compact minimal system** is a **compact system** (X, f) such that all orbits $\{f^n(x)\}_{n\in\mathbb{Z}}$ are dense.
Let (X, f) be a compact sysyem. A point $x \in X$ is **syndetically recurrent** if for every nbhd U of x there exists N, the set of return times $r(x, U) = \{n \in \mathbb{N} | f^n(x) \in U\}$ intersects with all N-blocks of consecutive natural numbers. Two equivalent definitions:

- All points in a minimal system is syndetically recurrent.
- Orbit closure of a syndetically recurrent point is minimal.

The space of compact minimal systems is a standard Borel space.

Minimal system

All compact metric space can be regarded as a closed subspace of the Hilbert cube. We can isomorphically embed (X, f) as a subshift of $(\mathbb{H}^{\mathbb{Z}}, S)$ by sending $x \in X$ to the point

$$(..., f^{-1}(x), x, f(x), ...).$$

We can view all compact minimal systems as a minimal subshift of $(\mathbb{H}^{\mathbb{Z}}, S)$.

Question

Definition

Two dynamical systems (X, f) and (Y, g) are **isomorphic** or **conjugate** if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, if h is just a continuous surjection we call h a **factor map**. Two systems are **flip conjugate** if (X, f) or (X, f^{-1}) are conjugate with (Y, g).

- What is the complexity of the conjugacy relation of minimal systems?
- Classifying all compact metric spaces admitting a minimal automorphism is a largely open problem.

Pointed minimal systems

Pointed minimal system is a minimal system together with a point, (X, f, x).
Two pointed minimal systems (X, f, x) and (Y, g, y) are isomorphic or conjugate if there is a isomorphism h between (X, f) and (Y, g) sending x to y.

Results

- (DGKKK) The conjugacy and flip conjugacy relations of Cantor minimal systems are not Borel.
- (Keya) Isomorphism of pointed Cantor minimal systems is Borel bireducible with $=_{\mathbb{R}}^{+}$.
- (Kaya) Isomorphism of pointed minimal systems is Borel.
- (P., Li) The conjugacy and flip conjugacy relations of minimal systems are not classifiable by countable structures.
- (P., Li) The conjugacy of pointed minimal systems is not classifiable by countable structures.

Toeplitz subshifts

- Let Σ be a compact metric space, we will look at subsystems of $(\Sigma^{\mathbb{Z}}, S)$.
- A sequence $x \in \Sigma^{\mathbb{Z}}$ is called a **Toeplitz sequence** if $x(n)$ is periodic for all n. **Toeplitz subshift** is the orbit closure of a Toelitz sequence.
- Let η be a Toeplitz sequence, every $x \in \overline{O}(\eta)$, the p-**skeleton** of x is

$$\mathrm{Per}_p(x) = \{n \in \mathbb{Z} | x(n + kp) = x(n) \forall k \in \mathbb{Z}\}.$$

  A period p of x is **essential** if $\mathrm{Per}_p(x) \neq \mathrm{Per}_q(x)$ for all $q < p$.

Factors of Toeplitz systems

- A system (X, T) is equicontinuous if $(T^n)_{n>0}$ are equicontinuous.
- All minimal systems admits a maximal equicontinuous factor which is unique up to isomorphism.
- The maximal equicontinuous factor of a Toplitz subshift is an Odometer system.
- For $\overline{O}(\eta)$, we can find an essential periodic structure (p_t) of it, such that $p_t | p_{t+1}$, all p_t are essential periods and $\cup_t \mathrm{Per}_{p_t}(\eta) = \mathbb{Z}$
- For all $x \in \overline{O}(\eta)$, x will have the same p_t-skeleton as $S^{n_t}\eta$ for $0 \leq n_t < p_t$. The map sending x to (n_t) is a factor map and $((p_t), 1)$ is the maximal equicontinuous factor of $\overline{O}(\eta)$.

Oxtoby subshift

- Let X be a compact metric space, (σ_i) be a sequence in X. Let (p_t) be a sequence of natural numbers such that
  1. $3 \leq p_1$.
  2. $3p_t \leq p_{t+1}$.
  3. $p_t \mid p_{t+1}$.
- We define the Oxtoby sequence η by induction. First, we fill all $\eta(kp_1 - 1)$ and $\eta(kp_1)$ with σ_1. In the $(n+1)^{th}$ step, we fill all empty positions in $[-p_n, p_n)$ by σ_{n+1} with period p_{n+1}.
- η is a well-defined Toeplitz sequence which was called an **Oxtoby sequence**.

Topological type of sequences

- Let X be a compact metric space, define an equivalence relation $E_{tt}(X)$ on X^ω:

$$(x_n)E_{tt}(y_n) \Leftrightarrow \forall (n_k)\ x_{n_k} \text{ converges iff } (y_{n_k}) \text{ converges.}$$

  And say two sequences have the same topological type.
- (X, f, x) and (Y, g, y) are isomorphic iff

$$(x, x, f(x), x, f^2(x), ...)\ E_{tt}(\mathbb{H}^\mathbb{Z})\ (y, y, g(y), y, g^2(y)...)$$

Outline of the proof

When (σ_i) and (σ_i') are not convergent.

Lemma

Suppose two Oxtoby subshifts $(\overline{O}(\eta), (p_i), (\sigma_i))$ *and* $(\overline{O}(\eta'), (p_i), (\sigma_i)')$ *are conjugate by* f*, then* $f(\eta) = S^n(\eta')$ *for some* n*.*

- Two Oxtoby subshifts with the same periodic sturcture are conjugate iff there is an isomorphism sending η to η'.
- $(S^n\eta)\ E_{tt}(\mathbb{H}^{\mathbb{Z}})\ (S^n\eta')$
- For Oxtoby sequences, this is equivalent to $(\sigma_i)E_{tt}(\mathbb{H})(\sigma_i')$.
- When both (σ_i) and (σ_i') converge, Oxtoby subshift is conjugate with Odometer (p_t).

Continue

- Let E_c, E_f, be the conjugacy and flip conjugacy relations of minimal systems. Let E_p be the conjugacy relation of pointed minimal systems.
- Starting with a sequence (σ_i) in $\mathbb{H}$. Choose a peridic structure of Oxtoby sequence p_t.
- Sending (σ_i) to $(\overline{O}(\eta), (p_i), (\sigma_i))$.
- By the previous Lemma, $E_{tt}(\mathbb{H})$ is Borel reducible to E_c and E_p.
- (Fact) Oxtoby subshift is conjugate to its inverse.
- $E_{tt}(\mathbb{H}) \leq_B E_f$.
- $E_{tt}(\mathbb{H}) \sim_B E_p$.

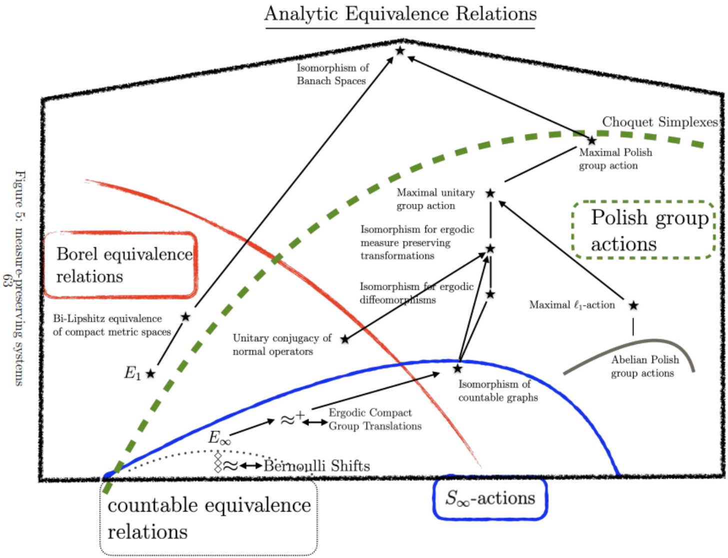
Last thing

- Let $c_0(S^1)$ be the space of sequences in S^1 converging to 0.
- Consider the orbit equivalence relation $E_{(S^1)^\omega}^{c_0(S^1)}$.
- By Hjorth's turbulent theorem, $E_{(S^1)^\omega}^{c_0(S^1)}$ is turbulent.
- Start with a sequence (s_n) in S^1, fix a countable dense subset (q_n) of S^1.
- Sending (s_n) to

$$(s_0, q_0, s_1, q_1, s_2, q_2....)$$

  is a Borel reduction from $E_{(S^1)^\omega}^{c_0(S^1)}$ to $E_{tt}(S^1)$.
- $E_{tt}(S^1) \leq_B E_{tt}(\mathbb{H})$ for obvious reasons.
- We are done!
- (P.,Li) Isomorphism relation of minimal systems is below a group action.

Updated picture



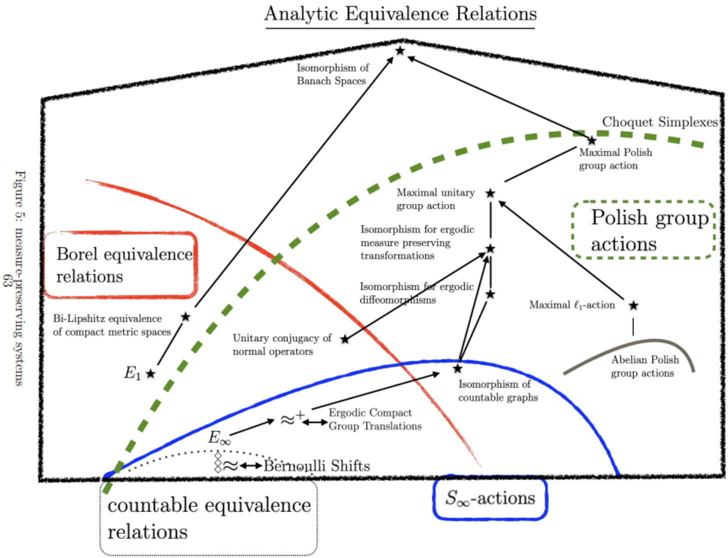
Sabok conjecture

- (Sabok) The affine homeomorphism relation of Choquet simplices is a complete orbit equivalence relation.
- (Zelinski) The affine homeomorphism relation of Bauer simplices is a complete orbit equivalence relation.
- (Williams) The invariant measure of oxtoby subshifts is affinely homeomorphic to $P(L(\sigma_i))$, where

$$L(\sigma_i) = \{\sigma | \lim_k \sigma_{n_k} = \sigma\}.$$

- (Downarowicz) All choquet simplces can be realized as invariant measures of a Toeplitz symbolic subshift.
- (Sabok conjuecture) Isomorphism relation of compact minimal systems is a complete orbit equivalence relation.

What we hope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Thanks!